03 Complex Numbers III

Calculator Free

1. [7 marks: 1, 3, 3] [TISC]

Given $u = 4\,cis\left(\dfrac{\pi}{3}\right)$ and $v = 2\,cis\left(\dfrac{k\pi}{3}\right)$ where k is a real number.

(a) If $2 \le k \le 4$, find $\dfrac{u}{v}$ in $r\,cis\,\theta$ form where $-\pi < \theta \le \pi$.

(b) Find $u \times v$ in $r\,cis\,\theta$ form where $2 \le k \le 4$ and $-\pi < \theta \le \pi$.

(c) Find k given that v is one of the square roots of u.

Calculator Free

2. [9 marks: 3, 3, 3] [TISC]

(a) Simplify $\dfrac{a^2\left[\cos\left(\dfrac{5\pi}{6}\right) - i\sin\left(\dfrac{5\pi}{6}\right)\right]}{4a\left[\cos\left(\dfrac{11\pi}{12}\right) + i\sin\left(\dfrac{11\pi}{12}\right)\right]}$, giving your answer in exact *cis* form.

(b) Simplify $\left[cis\left(\dfrac{\frac{\pi}{3}+2k\pi}{5}\right)\right]^5$, where $k = 0, 1, 2, 3, 4, 5, \ldots$.

Give your answer in exact *cis* form.

(c) Solve exactly for θ where $-\pi < \theta \leq \pi$ in

$(\cos\theta + i\sin\theta) \times (\cos\theta + i\sin\theta) = 1$.

Calculator Free

3. [6 marks: 3, 3] [TISC]

(a) Find n given that $\dfrac{1}{\cos 3\theta + i \sin 3\theta} = [cis\ \theta\ ]^n$

(b) Given that $\left| \dfrac{z-2}{z+2} \right| = 1$, where $z \neq 0$, show that z is completely imaginary.

4. [5 marks] [TISC]

Consider $z^5 = \dfrac{i}{32}$. Use De Moivre's Theorem to find all five roots of this equation. Show clearly how you obtained your answer.
Give your answer in *cis* form.

Calculator Free

5. [5 marks] [TISC]

Consider the equation $z^n = a + bi$. When plotted on an Argand diagram, two immediate adjacent roots are $cis\left(\frac{\pi}{12}\right)$ and $cis\left(\frac{7\pi}{12}\right)$. Find the value(s) of n, and corresponding exact values of a and b. Justify your answer.

6. [5 marks] [TISC]

Use De Moivre's Theorem to solve $z^4 = 1 + i$. Leave your answers in exact polar form. To obtain full marks for this question, you need to show how De Moivre's Theorem is used to obtain the answers.

Calculator Free

7. [9 marks: 4, 2, 3]

(a) If $z = \cos\theta + i\sin\theta$, show that $\cos n\theta = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right)$ and $\sin n\theta = \frac{1}{2i}\left(z^n - \frac{1}{z^n}\right)$.

(b) Hence, show that $\tan\theta = i\left(\frac{1-z^2}{1+z^2}\right)$.

(c) Use the result in (a) to prove that $\cos^2\theta - \sin^2\theta = \cos 2\theta$.

Calculator Free

8. [8 marks: 5, 3] [TISC]

(a) Use De Moivre's theorem to show that

$$\cos(6\theta) = -1 + 18\cos^2(\theta) - 48\cos^4(\theta) + 32\cos^6(\theta)\,.$$

The following identities may be useful.

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

$$\sin^6\theta = 1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta$$

$$\sin^4\theta = 1 - 2\cos^2\theta + \cos^4\theta$$

(b) Hence, or otherwise, find the exact roots of the equation

$$32x^6 - 48x^4 + 18x^2 - 1 = 0.$$

Calculator Free

9. [7 marks: 3, 2, 2] [TISC]

Let $w = z + \frac{1}{z}$.

It can be shown that $w^3 + w^2 - 2w - 2 = \left(z^3 + \frac{1}{z^3}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z + \frac{1}{z}\right)$.

Given that $z = cis\ \theta$, a commonly used result is $z^n + \frac{1}{z^n} = 2\cos n\theta$.

(a) Show that solving $w^3 + w^2 - 2w - 2 = 0$ is equivalent to solving $\cos 3\theta + \cos 2\theta + \cos\theta = 0$.

(b) The solutions to $w^3 + w^2 - 2w - 2 = 0$ are $-\sqrt{2}$, -1 and $\sqrt{2}$.

Explain clearly why the solution $w = -1$ implies that $z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

(c) Hence, find one solution to $\cos 3\theta + \cos 2\theta + \cos\theta = 0$.

Calculator Assumed

10. [13 marks: 4, 2, 2, 5] [TISC]

Let $z = cis\,\theta$.

(a) Prove that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$.

(b) If $w = z + \dfrac{1}{z}$, prove that

$$w^3 + w^2 - 2w - 2 = \left(z + \frac{1}{z}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z^3 + \frac{1}{z^3}\right).$$

Calculator Assumed

10. (c) Use parts (a) and (b) to show that the equation $\cos\theta + \cos 2\theta + \cos 3\theta = 0$ can be rewritten as $w^3 + w^2 - 2w - 2 = 0$.

(d) Given that $-\pi < \theta \leq \pi$, use part (c) to solve for θ where

$$\cos\theta + \cos 2\theta + \cos 3\theta = 0.$$

Calculator Assumed

11. [9 marks: 2, 2, 3, 2]

(a) Let $z_0 = 2\,cis\left(\frac{\pi}{5}\right)$.

(i) Show that $z_0^{\,5} = -32$.

(ii) Hence, find four other complex numbers in polar form where $-\pi < \theta \leq \pi$ such that $z^5 = -32$.

(b) Determine $cis\left(\frac{\theta}{4}\right) + cis\left(-\frac{\theta}{4}\right)$ in the form $a + bi$.

(c) Use your answer in (b) to prove that $2\,cis\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{4}\right) = 1 + cis\left(\frac{\theta}{2}\right)$.

Calculator Assumed

12. [7 marks: 4, 3] [TISC]

(a) Use de Moivre's Theorem to solve the equation $z^4 + 16 = 0$ where z is a complex number. Give your answer in *cis* form.

(b) Use your answer in (a) to factorise $z^4 + 16$.

03 Complex Numbers III

Calculator Free

1. [7 marks: 1, 3, 3] [TISC]

Given $u = 4\,cis\left(\frac{\pi}{3}\right)$ and $v = 2\,cis\left(\frac{k\pi}{3}\right)$ where k is a real number.

(a) If $2 \le k \le 4$, find $\frac{u}{v}$ in $r\,cis\,\theta$ form where $-\pi < \theta \le \pi$.

$$\frac{u}{v} = 2\,cis\left(\frac{\pi}{3} - \frac{k\pi}{3}\right) \quad ✓$$

(b) Find $u \times v$ in $r\,cis\,\theta$ form where $2 \le k \le 4$ and $-\pi < \theta \le \pi$.

$uv = 8\,cis\left(\frac{\pi}{3} + \frac{k\pi}{3}\right)$ ✓

$= 8\,cis\left(\frac{\pi(k+1)}{3}\right)$ ✓

$= 8\,cis\left(\frac{\pi(k+1)}{3} - 2\pi\right)$ ✓

as $\left(\frac{\pi(k+1)}{3}\right)$ is outside the principal domain for $2 \le k \le 4$.

(c) Find k given that v is one of the square roots of u.

$u = 4\,cis\left(\frac{\pi}{3}\right)$.

$\sqrt{u} = 2\,cis\left(\frac{\frac{\pi}{3} + 2n\pi}{2}\right)$ ✓

$= 2\,cis\left(\frac{\pi}{6}\right)$ or $2\,cis\left(\frac{7\pi}{6}\right)$

$= 2\,cis\left(\frac{\pi}{6}\right)$ or $2\,cis\left(\frac{-5\pi}{6}\right)$

Hence, $k = \frac{1}{2}$ or $\frac{-5}{2}$. ✓✓

Calculator Free

2. [9 marks: 3, 3, 3] [TISC]

(a) Simplify $\dfrac{a^2\left[\cos\left(\frac{5\pi}{6}\right) - i\sin\left(\frac{5\pi}{6}\right)\right]}{4a\left[\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right]}$, giving your answer in exact *cis* form.

$$\frac{a^2\left[\cos\left(\frac{5\pi}{6}\right) - i\sin\left(\frac{5\pi}{6}\right)\right]}{4a\left[\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right]} = \frac{a^2 cis\left(-\frac{5\pi}{6}\right)}{4a\,cis\left(\frac{11\pi}{12}\right)} \quad ✓$$

$= \frac{a}{4} cis\left(-\frac{7\pi}{4}\right)$ ✓

$= \frac{a}{4} cis\left(\frac{\pi}{4}\right)$ or $-\frac{a}{4} cis\left(-\frac{3\pi}{4}\right)$ ✓

(b) Simplify $\left[cis\left(\dfrac{\frac{\pi}{3} + 2k\pi}{5}\right)\right]^5$, where $k = 0, 1, 2, 3, 4, 5, \ldots$.

Give your answer in exact *cis* form.

$\left[cis\left(\frac{\frac{\pi}{3}+2k\pi}{5}\right)\right]^5 = \left[cis\left(5 \times \frac{\frac{\pi}{3}+2k\pi}{5}\right)\right]$ ✓

$= cis\left(\frac{\pi}{3} + 2k\pi\right)$ ✓

$= cis\left(\frac{\pi}{3}\right) \times cis(2k\pi)$

$= cis\left(\frac{\pi}{3}\right)$ ✓

(c) Solve <u>exactly</u> for θ where $-\pi < \theta \le \pi$ in
$(\cos\theta + i\sin\theta) \times (\cos\theta + i\sin\theta) = 1$.

$(\cos\theta + i\sin\theta) \times (\cos\theta + i\sin\theta) = 1$

$\Rightarrow cis\,\theta \times cis\,\theta = 1$

$cis\,2\theta = 1$ ✓

$2\theta = 0, 2\pi$ ✓

$\Rightarrow \theta = 0, \pi$ ✓

Calculator Free

3. [6 marks: 3, 3] [TISC]

(a) Find n given that $\dfrac{1}{\cos 3\theta + i \sin 3\theta} = [cis\ \theta]^n$

$$\frac{1}{\cos 3\theta + i\sin 3\theta} = cis\ 0 - cis\ 3\theta \quad ✓$$
$$= cis\ (-3\theta)$$
$$= [cis\ \theta]^{-3} \quad ✓$$
Hence, $n = -3$. ✓

(b) Given that $\left|\dfrac{z-2}{z+2}\right| = 1$, where $z \neq 0$, show that z is completely imaginary.

Let $z = x + iy$.

$\left|\frac{z-2}{z+2}\right| = 1 \Rightarrow |z-2| = |z+2|$ ✓

$(x-2)^2 + y^2 = (x+2)^2 + y^2$ ✓

$x^2 - 4x + 4 = x^2 + 4x + 4$

$x = 0$ ✓

Hence, $z = iy$ which is completely imaginary.

4. [5 marks] [TISC]

Consider $z^5 = \dfrac{i}{32}$. Use De Moivre's Theorem to find all five roots of this equation. Show clearly how you obtained your answer.
Give your answer in *cis* form.

$z^5 = \left(\frac{1}{2}\right)^5 cis\left(\frac{\pi}{2} + 2n\pi\right)$ ✓

$z = \left(\frac{1}{2}\right) cis\left(\dfrac{\frac{\pi}{2} + 2n\pi}{5}\right)$ ✓

Hence,

$z = \left(\frac{1}{2}\right) cis\left(\frac{\pi}{10}\right),$ ✓

$\left(\frac{1}{2}\right) cis\left(\frac{\pi}{2}\right),$

$\left(\frac{1}{2}\right) cis\left(\frac{9\pi}{10}\right),$ ✓

$\left(\frac{1}{2}\right) cis\left(\frac{13\pi}{10}\right) = \left(\frac{1}{2}\right) cis\left(-\frac{7\pi}{10}\right),$

$\left(\frac{1}{2}\right) cis\left(\frac{17\pi}{10}\right) = \left(\frac{1}{2}\right) cis\left(-\frac{3\pi}{10}\right).$ ✓

Calculator Free

5. [5 marks] [TISC]

Consider the equation $z^n = a + bi$. When plotted on an Argand diagram, two immediate adjacent roots are $cis\left(\dfrac{\pi}{12}\right)$ and $cis\left(\dfrac{7\pi}{12}\right)$. Find the value(s) of n, and corresponding exact values of a and b. Justify your answer.

Angular difference between roots $= \frac{\pi}{2}$. ✓

Hence, number of roots $= \dfrac{2\pi}{\left(\frac{\pi}{2}\right)} = 4$. ✓

As the roots are adjacent and immediate,

$\Rightarrow n = 4$ ✓

$\Rightarrow a = \frac{1}{2}, b = \frac{\sqrt{3}}{2}$ ✓✓

6. [5 marks] [TISC]

Use De Moivre's Theorem to solve $z^4 = 1 + i$. Leave your answers in exact polar form. To obtain full marks for this question, you need to show how De Moivre's Theorem is used to obtain the answers.

$z^4 = \sqrt{2}\ cis\ \frac{\pi}{4}$

$z = \left(\sqrt{2}\ cis\ \frac{\pi}{4}\right)^{1/4} = 2^{1/8} cis\left(\dfrac{\frac{\pi}{4} + 2n\pi}{4}\right)$ ✓

$\Rightarrow z = 2^{1/8} cis\left(\frac{\pi}{16}\right),$ ✓

$2^{1/8} cis\left(\frac{9\pi}{16}\right),$ ✓

$2^{1/8} cis\left(-\frac{15\pi}{16}\right),$ ✓

$2^{1/8} cis\left(-\frac{7\pi}{16}\right)$ ✓

Calculator Free

7. [9 marks: 4, 2, 3]

(a) If $z = \cos\theta + i\sin\theta$, show that $\cos n\theta = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right)$ and $\sin n\theta = \frac{1}{2i}\left(z^n - \frac{1}{z^n}\right)$.

$z^n = (\cos\theta + i\sin\theta)^n$
$= \cos n\theta + i\sin n\theta$ I ✓
$z^{-n} = (\cos\theta + i\sin\theta)^{-n}$
$= \cos(-n\theta) + i\sin(-n\theta)$
$= \cos n\theta - i\sin n\theta$ II ✓

I + II $\left(z^n + \frac{1}{z^n}\right) = 2\cos n\theta$ ✓

$\Rightarrow \cos n\theta = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right)$

I − II $\left(z^n - \frac{1}{z^n}\right) = 2i\sin n\theta$ ✓

$\Rightarrow \sin n\theta = \frac{1}{2i}\left(z^n - \frac{1}{z^n}\right)$

(b) Hence, show that $\tan\theta = i\left(\frac{1-z^2}{1+z^2}\right)$.

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$= \frac{\frac{1}{2i}\left[z - \frac{1}{z}\right]}{\frac{1}{2}\left[z + \frac{1}{z}\right]} = \frac{\left[z - \frac{1}{z}\right]}{i\left(z + \frac{1}{z}\right)}$ ✓

$= -i\left(\frac{z^2-1}{z^2+1}\right) = i\left(\frac{1-z^2}{1+z^2}\right)$ ✓

(c) Use the result in (a) to prove that $\cos^2\theta - \sin^2\theta = \cos 2\theta$.

LHS $= \cos^2\theta - \sin^2\theta$

$= \left[\frac{1}{2}\left(z + \frac{1}{z}\right)\right]^2 - \left[\frac{1}{2i}\left(z - \frac{1}{z}\right)\right]^2$ ✓✓

$= \left[\frac{1}{4}\left(z^2 + 2 + \frac{1}{z^2}\right)\right] - \left[-\frac{1}{4}\left(z^2 - 2 + \frac{1}{z^2}\right)\right]$

$= \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right)$ ✓

$= \cos 2\theta =$ RHS

Calculator Free

8. [8 marks: 5, 3] [TISC]

(a) Use De Moivre's theorem to show that

$$\cos(6\theta) = -1 + 18\cos^2(\theta) - 48\cos^4(\theta) + 32\cos^6(\theta).$$

The following identities may be useful.

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$
$$\sin^6\theta = 1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta$$
$$\sin^4\theta = 1 - 2\cos^2\theta + \cos^4\theta$$

$(\text{cis}\,\theta)^6 = (\cos\theta + i\sin\theta)^6$

$\cos 6\theta + i\sin 6\theta = \cos^6\theta + 6\cos^5\theta\,(i\sin\theta) + 15\cos^4\theta\,(i\sin\theta)^2$
$+ 20\cos^3\theta\,(i\sin\theta)^3 + 15\cos^2\theta\,(i\sin\theta)^4$
$+ 6\cos\theta\,(i\sin\theta)^5 + (i\sin\theta)^6$ ✓✓

Equate real part:

$\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$ ✓
$= \cos^6\theta - 15\cos^4\theta\,(1-\cos^2\theta) + 15\cos^2\theta\,(1 - 2\cos^2\theta + \cos^4\theta)$
$-(1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta)$ ✓
$= -1 + 18\cos^2\theta - 48\cos^4\theta + 32\cos^6\theta.$ ✓

(b) Hence, or otherwise, find the exact roots of the equation

$$32x^6 - 48x^4 + 18x^2 - 1 = 0.$$

Let $x = \cos\theta$.
Hence, equation becomes:
$32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1 = 0$ ✓

But $32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1 = \cos 6\theta$.
Hence, $\cos 6\theta = 0$.

$\Rightarrow 6\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}$ ✓

$\theta = \frac{\pi}{12}, \frac{3\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{9\pi}{12}, \frac{11\pi}{12}$

Therefore, solutions are:
$x = \cos\frac{\pi}{12}, \cos\frac{3\pi}{12}, \cos\frac{5\pi}{12}, \cos\frac{7\pi}{12}, \cos\frac{9\pi}{12}, \cos\frac{11\pi}{12}$ ✓

Calculator Free

9. [7 marks: 3, 2, 2] [TISC]

Let $w = z + \frac{1}{z}$.

It can be shown that $w^3 + w^2 - 2w - 2 = \left(z^3 + \frac{1}{z^3}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z + \frac{1}{z}\right)$.

Given that $z = cis\ \theta$, a commonly used result is $z^n + \frac{1}{z^n} = 2\cos n\theta$.

(a) Show that solving $w^3 + w^2 - 2w - 2 = 0$ is equivalent to solving $\cos 3\theta + \cos 2\theta + \cos\theta = 0$.

$w^3 + w^2 - 2w - 2 = \left(z^3 + \frac{1}{z^3}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z + \frac{1}{z}\right)$ ✓

$= 2\cos 3\theta + 2\cos 2\theta + 2\cos\theta$. ✓✓

Hence, $w^3 + w^2 - 2w - 2 = 0$ is equivalent to $2\cos 3\theta + 2\cos 2\theta + 2\cos\theta = 0$.

(b) The solutions to $w^3 + w^2 - 2w - 2 = 0$ are $-\sqrt{2}$, -1 and $\sqrt{2}$.

Explain clearly why the solution $w = -1$ implies that $z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

$w = -1 \Rightarrow \left(z + \frac{1}{z}\right) = -1$ ✓

$z^2 + z + 1 = 0$ ✓

$z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

(c) Hence, find one solution to $\cos 3\theta + \cos 2\theta + \cos\theta = 0$.

One solution to $w^3 + w^2 - 2w - 2 = 0$ is $w = -1$.

$w = -1$ is equivalent to $z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

Since $z = cis\ \theta$: $cis\ \theta = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ ✓

$\cos\theta + i\sin\theta = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

$\Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}$

and $\sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{2\pi}{3}$

Hence, $\theta = \frac{2\pi}{3}$. ✓

Calculator Assumed

10. [13 marks: 4, 2, 2, 5] [TISC]

Let $z = cis\ \theta$.

(a) Prove that $z^n + \frac{1}{z^n} = 2\cos n\theta$.

$z = cis\ \theta \Rightarrow z^n = cis\ n\theta$ and $\frac{1}{z^n} = cis\ (-n\theta)$ ✓

$\text{LHS} = z^n + \frac{1}{z^n}$

$= cis\ n\theta + cis\ (-n\theta)$ ✓

$= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta + i\sin n\theta + \cos(n\theta) - i\sin(n\theta)$ ✓ ✓

$= 2\cos n\theta = \text{RHS}$

(b) If $w = z + \frac{1}{z}$, prove that

$$w^3 + w^2 - 2w - 2 = \left(z + \frac{1}{z}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z^3 + \frac{1}{z^3}\right).$$

$\text{LHS} = w^3 + w^2 - 2w - 2$

$= \left(z + \frac{1}{z}\right)^3 + \left(z + \frac{1}{z}\right)^2 - 2\left(z + \frac{1}{z}\right) - 2$ ✓

$= z^3 + z^2 + z + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3}$

$= \left(z + \frac{1}{z}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z^3 + \frac{1}{z^3}\right)$ ✓

$= \text{RHS}$

```
expand((x+1/x)^3+(x+1/x)^2-2(x+1/x)-2)
                    z^3+z^2+z+1/z+1/z^2+1/z^3
```

OR

```
x+1/x▸w
                                    z+1/z
expand(w^3+w^2-2w-2)
                    z^3+z^2+z+1/z+1/z^2+1/z^3
```

Calculator Assumed

10. (c) Use parts (a) and (b) to show that the equation $\cos\theta + \cos 2\theta + \cos 3\theta = 0$ can be rewritten as $w^3 + w^2 - 2w - 2 = 0$.

From (a): $2\cos\theta = \left(z + \frac{1}{z}\right)$, $2\cos 2\theta = \left(z^2 + \frac{1}{z^2}\right)$ and $2\cos 3\theta = \left(z^3 + \frac{1}{z^3}\right)$

Hence: $\cos\theta + \cos 2\theta + \cos 3\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) + \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right) + \frac{1}{2}\left(z^3 + \frac{1}{z^3}\right)$

Therefore: $\cos\theta + \cos 2\theta + \cos 3\theta = 0$

$\Rightarrow \frac{1}{2}\left(z + \frac{1}{z}\right) + \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right) + \frac{1}{2}\left(z^3 + \frac{1}{z^3}\right) = 0$ ✓

$\left(z + \frac{1}{z}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z^3 + \frac{1}{z^3}\right) = 0$

But from (b): $\left(z + \frac{1}{z}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z^3 + \frac{1}{z^3}\right) = w^3 + w^2 - 2w - 2$ ✓

Hence: $\cos\theta + \cos 2\theta + \cos 3\theta = 0$

is equivalent to $w^3 + w^2 - 2w - 2 = 0$

(d) Given that $-\pi < \theta \le \pi$, use part (c) to solve for θ where $\cos\theta + \cos 2\theta + \cos 3\theta = 0$.

From (c): $\cos\theta + \cos 2\theta + \cos 3\theta = 0$ I

is equivalent to $w^3 + w^2 - 2w - 2 = 0$

Solving for w: $w = -1, \pm\sqrt{2}$ ✓

But $w = z + \frac{1}{z}$ $\Rightarrow z + \frac{1}{z} = -1, \pm\sqrt{2}$ ✓

But $z + \frac{1}{z} = 2\cos\theta$ $\Rightarrow 2\cos\theta = -1, \pm\sqrt{2}$

$\cos\theta = -\frac{1}{2}, \pm\frac{\sqrt{2}}{2}$ ✓

Hence solution for (I) is: $\theta = \pm\frac{2\pi}{3}, \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$ ✓✓

Calculator Assumed

11. [9 marks: 2, 2, 3, 2]

(a) Let $z_0 = 2\,cis\left(\frac{\pi}{5}\right)$.

(i) Show that $z_0^{\,5} = -32$.

$z_0^{\,5} = \left[2\,cis\left(\frac{\pi}{5}\right)\right]^5 = 2^5\,cis\left(\frac{\pi}{5} \times 5\right)$ ✓

$= 32\,cis(\pi) = -32$ ✓

(ii) Hence, find four other complex numbers in polar form where $-\pi < \theta \le \pi$ such that $z^5 = -32$.

$z_1 = 2\,cis\left(\frac{\pi}{5} + \frac{2\pi}{5}\right) = 2\,cis\left(\frac{3\pi}{5}\right)$

$z_2 = 2\,cis\left(\frac{\pi}{5} + \frac{4\pi}{5}\right) = 2\,cis(\pi)$

$z_3 = 2\,cis\left(\frac{\pi}{5} + \frac{6\pi}{5}\right) = 2\,cis\left(-\frac{3\pi}{5}\right)$

$z_4 = 2\,cis\left(\frac{\pi}{5} + \frac{8\pi}{5}\right) = 2\,cis\left(-\frac{\pi}{5}\right)$ ✓✓

(b) Determine $cis\left(\frac{\theta}{4}\right) + cis\left(-\frac{\theta}{4}\right)$ in the form $a + bi$.

$cis\left(\frac{\theta}{4}\right) + cis\left(-\frac{\theta}{4}\right) = [\cos\left(\frac{\theta}{4}\right) + i\sin\left(\frac{\theta}{4}\right)] + [\cos\left(-\frac{\theta}{4}\right) + i\sin\left(-\frac{\theta}{4}\right)]$ ✓

$= [\cos\left(\frac{\theta}{4}\right) + i\sin\left(\frac{\theta}{4}\right)] + [\cos\left(\frac{\theta}{4}\right) - i\sin\left(\frac{\theta}{4}\right)]$ ✓

$= 2\cos\left(\frac{\theta}{4}\right) + 0\,i$ ✓

(c) Use your answer in (b) to prove that $2\,cis\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{4}\right) = 1 + cis\left(\frac{\theta}{2}\right)$.

LHS $= 2\,cis\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{4}\right)$

$= cis\left(\frac{\theta}{4}\right) \times [cis\left(\frac{\theta}{4}\right) + cis\left(-\frac{\theta}{4}\right)]$ ✓

$= cis\left(\frac{\theta}{2}\right) + cis\,0$ ✓

$= cis\left(\frac{\theta}{2}\right) + 1 =$ RHS

Calculator Assumed

12. [7 marks: 4, 3] [TISC]

(a) Use de Moivre's Theorem to solve the equation $z^4 + 16 = 0$ where z is a complex number. Give your answer in *cis* form.

$z^4 = -16$

$z^4 = 16\ cis\ (\pi + 2n\pi)$

$z = [16 cis (\pi + 2n\pi)]^{\frac{1}{4}}$ ✓

$z = 2\ cis\left(\frac{\pi + 2n\pi}{4}\right)$ ✓

$z = 2\ cis\left(\frac{\pi}{4}\right), 2\ cis\left(\frac{3\pi}{4}\right),$ ✓

$2\ cis\left(-\frac{3\pi}{4}\right), 2\ cis\left(-\frac{\pi}{4}\right)$ ✓

(b) Use your answer in (a) to factorise $z^4 + 16$.

$z = 2\ cis\left(\frac{\pi}{4}\right), 2\ cis\left(\frac{3\pi}{4}\right), 2\ cis\left(-\frac{3\pi}{4}\right), 2\ cis\left(-\frac{\pi}{4}\right)$

$= \sqrt{2}\,(1 + i), \sqrt{2}\,(-1 + i), \sqrt{2}\,(-1 - i), \sqrt{2}\,(1 - i)$ ✓

Hence $z^4 + 16 = [z - \sqrt{2}\,(1 + i)]\,[z - \sqrt{2}\,(-1 + i)]\,[z - \sqrt{2}\,(-1 - i)]\,[z - \sqrt{2}\,(1 - i)]$ ✓✓

04 The Factor & Remainder Theorems

Calculator Free

1. [11 marks: 3, 3, 5]

(a) Prove that if $(x - a)^2$ is a factor of the real polynomial $f(x)$, then $(x - a)$ is a factor of $f'(x)$ where $f'(x)$ is the derivative of $f(x)$ with respect to x.

If $(x - a)^2$ is a factor of $f(x)$, then

$f(x) = (x - a)^2 \times Q(x)$. ✓

$f'(x) = 2(x - a) \times Q(x) + (x - a)^2 \times Q'(x)$ ✓

$f'(a) = 2(a - a) \times Q(a) + (a - a)^2 \times Q'(a)$

$= 0$ ✓

Hence, $(x - a)$ is a factor of $f'(x)$

(b) $(2x - 1)^2$ is a factor of $4x^4 - kx^3 - 3x^2 + kx - 1$. Determine the value of k.

Let $f(x) = 4x^4 - kx^3 - 3x^2 + kx - 1$

$f\left(\frac{1}{2}\right) = 0 \Rightarrow \frac{1}{4} - \frac{k}{8} - \frac{3}{4} + \frac{k}{2} - 1 = 0$ ✓✓

$\frac{3k}{8} = \frac{3}{2}$

$k = 4$ ✓

(c) $(x + 2)^2$ is a factor of $2x^4 + ax^3 + bx^2 - 4$. Determine the values of a and b.

Let $f(x) = 2x^4 + ax^3 + bx^2 - 4$

$f(-2) = 0 \Rightarrow 32 - 8a + 4b - 4 = 0$

$2a - b = 7$ I ✓

$f'(x) = 8x^3 + 3ax^2 + 2bx$ ✓

$f'(-2) = -64 + 12a - 4b = 0$

$3a - b = 16$ II ✓

II − I $a = 9$ ✓

$b = 11$ ✓